# Shape of a Cracking Whip

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The crack of a whip is produced by a shock wave created by the supersonic motion of the tip of the whip in the air. A simple dynamical model for the propagation and acceleration of waves in the motion of whips is presented. The respective contributions of tension, tapering, and boundary conditions in the acceleration of an initial impulse are studied theoretically and numerically.

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Whips are among the most misunderstood and misrepresented objects in today's culture. Whips are usually thought of either as weapons (like the ones used by Zorro and Indiana Jones) or as instruments of torture associated with slavery and perverse activities. The world of whips is divided into two main categories, the "pain-making" whips which are short, bulky, and often multithreaded (such as the infamous cat-o'-nine-tails) and the "noise-making" whips which are long, tapered, and single threaded (such as the stockwhip or the bullwhip). It is the latter category which is discussed in this Letter. Despite their frightening appearance, the noise-making whips have never been seriously used as weapons or torture devices; their only purpose is to produce, under expert manipulation, a very distinct, loud crack. Historically, they were primarily used to direct animals (such as horses and cattle) and secondarily used as a means of communication. Since the introduction of cars and the development of modern ranching in the early 20th century, the practical use of whips has become almost extinct and the tradition of whips survives only at the hands of fine leather craftspeople and a few whip enthusiasts for entertainment purposes [1]. In this Letter we focus our attention on the physical mechanism responsible for the crack.

It is usually believed that the whip crack is a sonic boom produced when the tip reaches supersonic speed. This explanation, part of most physicist trivia questions, was first advanced by Lumer in 1905 and substantiated by Prandtl in 1912. The first experiment on whips was performed by Carrière in 1927 [2]. He showed through high-speed shadow photography that a sonic boom is associated with the crack of the whip. Further observations were recorded by Bernstein, Hall, and Trent in 1958 [3]. More recently, Krehl, Engemann, and Schwenkel performed a beautiful high-speed digital photography experiment in [4] where accelerations of up to $50\,000g$ were recorded (Fig. 1). Moreover, they report a rather puzzling observation: the sonic boom is emitted when the tip velocity reaches about *twice* the speed of sound in the air.

At the theoretical level, whip dynamics is one of these many problems of physics that we take for granted. Different simple kinematic models [5] have been proposed, all based on the same central dogma of energy conservation already put forward by Bragg in 1920 [6]: "The crack of the whip is no doubt due to the same cause (shock wave): a wave runs down the cord and carries energy to the lash at the end, which is so light that it is set into most violent motion." More precisely, the argument goes as follows: when the whip is made to crack, it is given an initial velocity and the moving part of the whip gets localized in an increasingly smaller section of the whip close to the tip. If the energy is conserved, the velocity of this section must increase. Eventually, as the length of this section decreases to zero, the end part of the tip moves with unbounded velocity and cracks as soon as it reaches the velocity of the sound in the air. If the whip is tapered, this effect should be further enhanced as the mass of the moving part decreases even faster. However, Steiner and Troger [7] point out that if one assumes that linear momentum is conserved rather than energy, then the tip of the whip travels only at its initial velocity. Other authors [8] attribute the acceleration of the tip to the conservation of angular momentum (not unlike the acceleration given to a golf club or the dinosaur tail when swung). The main problem is that these models are based on kinematics; the motion and shape of the whip is assumed and the velocity is deduced from a given conservation principle. Not surprisingly, they cannot satisfy different conservation laws

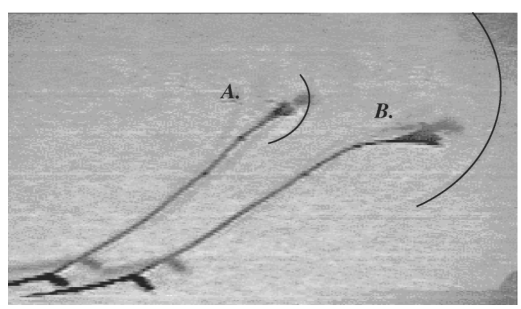


FIG. 1. High-speed digital shadowgraphs of a cracking whip and its sonic boom. The time interval between the two pictures is 111 $\mu$s. The solid lines are superimposed over the shock waves. The tip velocity at the time of the crack was Mach 2. Picture courtesy of Krehl *et al.* [4].

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simultaneously and do not provide insight into the shape of the whip.

The purpose of this Letter is to reconsider the dynamics of the whip and reconcile these seemingly contradictory aspects such as the relationship between sonic boom and tip velocity, the effect of tapering, the boundary conditions, and the role of energy, linear momentum, and angular momentum. To do so, we must first understand the way whips are made and their actual motion in the air. The first part of the whip (the "handle" and sometimes a "swivel") is designed to facilitate the formation of the initial shape (see below), whereas the second part, the "thong," is a long tapered filament made to carry and accelerate this shape to a small string, the "cracker." A well-designed thong has very low dissipation and good whips are tested by imparting a small deformation to a whip on the ground to see it run to the end with little loss. The thong is finely cross-braided (kangaroo leather is preferred) so that an initially planar shape remains planar [9]. There are two principal methods to crack a whip that can be performed in different planes (laterally along the body, sideways, over the head, and so on) [10]. The first one, the downward snap, consists in moving the handle vertically as to move the entire whip up and then suddenly move the handle down to invert the velocity of the whip and localize the moving part to the tip. While this is the most intuitive way to crack a whip, it is not as efficient as the forward crack, where the handle is moved in the air as to form an initial loop that propagates along the whip to the end (see Fig. 1). The formation and propagation of a traveling loop on an elastic filament is also observed in other contexts such as the roll cast in fly-fishing [11] or the looping of flagella in sperm motility [12].

We focus our attention on the acceleration and localization of an initial loop in a tapered whip. In light of the previous remarks, we model the whip as a planar unshearable and inextensible elastic rod with homogeneous density, circular cross section of varying radius $R$, and linear constitutive relationship. Here, we neglect the effect of gravity (the same crack can be performed in vertical or horizontal planes) and the friction of the whip in the air (crucial to explain the creation of a sonic boom but not for the acceleration of the loop itself). Let $\mathbf{x}(s,t) = x(s,t)\mathbf{e}_x + y(s,t)\mathbf{e}_y$ be the centerline of the rod in the $x$-$y$ plane, where $s$ and $t$ are arclength and time [with derivatives denoted by $()'$ and $(\dot{})$]. The curvature of the rod is given by $\kappa = \varphi'$, where $\varphi$ is the angle between the $x$ axis and the tangent vector $\mathbf{t} = \mathbf{x}' = \cos(\varphi)\mathbf{e}_x + \sin(\varphi)\mathbf{e}_y$. The balance of linear and angular momenta, together with the constitutive relationship that relates the angle $\varphi$ to the moment, provides a set of equations for the position of the rod $(x, y)$ and the force $\mathbf{F} = F\mathbf{e}_x + G\mathbf{e}_y$ [13]:

$$\delta\ddot{x} = F', \qquad \delta\ddot{y} = G', \tag{1}$$

$$\delta^2\ddot{\varphi} = (\delta^2\varphi')' + G\cos\varphi - F\sin\varphi. \tag{2}$$

These equations and the variables have been rescaled so that force and position are dimensionless, the radius is equal to 2 at a reference point, the speed of a sound wave in the material is equal to 1, and $\delta = R^2(s)/R^2(0)$. The velocity of sound in leather is 220 m/s [14], comparable to the propagation of sound in the air (330 m/s). The conservation law, $\dot{\mathcal{H}} = \mathcal{W}'$, with

$$\mathcal{H} = \frac{\delta}{2}[\delta(\dot{\varphi}^2 + \varphi'^2) + \dot{x}^2 + \dot{y}^2],$$

$$\mathcal{W} = \delta^2\varphi'\dot{\varphi} + F\dot{x} + G\dot{y},$$

is associated to the energy $H = \int \mathcal{H}\, ds$. When the radius is kept constant and an infinite rod is considered, these equations support a two-parameter family of traveling loops [15] (see Fig. 2) given by $x_l = s - 2\alpha\tanh(\xi/\alpha)$, and $y_l = 2\alpha\,\mathrm{sech}(\xi/\alpha)$, where $\xi = s - ct$. The first interesting observation is that the velocity of a material point at the top of the loop is exactly twice the velocity of the loop itself (see Fig. 2). This simple general kinematic property of traveling loops together with an analysis of the point of origin of the shock wave in Fig. 1 suggests a simple explanation for the experimental observation of the tip velocity reported by Krehl *et al.*: the shock wave is emitted by the loop rather than by the tip of the whip. When the loop reaches supersonic velocity and cracks in the air, the tip of the whip has twice the velocity of the loop. The actual interaction of the supersonic cracker in the air is an extremely complex phenomenon which could be understood only by analyzing the motion of a movable accelerating elastic boundary in a supersonic flow.

A loop of height $2\alpha$ and speed $c$ has an energy $H_l$ and an end tension $T_l$ given by

$$H_l = 4\delta(\delta + \delta c^2 + \alpha^2 c^2)/\alpha, \tag{3}$$

$$T_l = \delta(\delta - \delta c^2 + \alpha^2 c^2)/\alpha^2. \tag{4}$$

We use the solution $(x_l, y_l)$ as an initial profile. The first problem that we address is the effect of tapering on the loop shape and velocity. One might be tempted, following the aforementioned energy argument, to expect the loop to accelerate on a tapering rod. However, a quick look at (3) reveals that the energy would also be conserved if

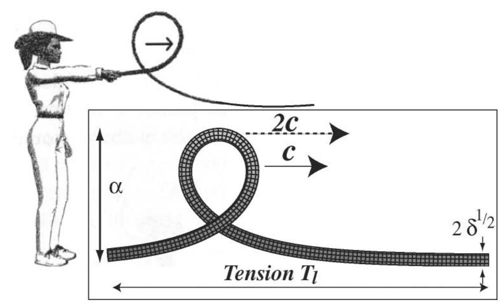


FIG. 2. Sketch of a forward crack (drawing courtesy of Conway [10]). Traveling loop solution of the planar equation $(x_l, y_l)$. The loop travels at velocity $c$, whereas a material point on top of the loop moves at velocity $2c$.

the loop slows down and shrinks. Clearly, the kinematic argument fails here. In general, one cannot assume a given asymptotic shape for the whip and deduce its velocity.

To isolate the effect of tapering, we consider an infinite rod with a slowly varying cross section $\delta = \delta(\epsilon s)$ with $\epsilon \ll 1$ and compute asymptotically [16] the change in height and velocity of a loop of initial height and velocity $[c_i = c(\delta = 1),\, \alpha_i = \alpha(\delta = 1)]$ by introducing the long scales: $S = \epsilon s$, $\xi = \frac{1}{\epsilon}\int_0^S A(s)\, ds - t$. We rewrite the conservation law $\dot{\mathcal{H}} = \mathcal{W}'$ in terms of the new variables and expand it in powers of $\epsilon$. The compatibility condition to first order in $\epsilon$ yields $\partial_S(\int_{-\infty}^{\infty} \mathcal{W}_0\, d\xi) = 0$, where $\mathcal{W}_0$ is the work density $\mathcal{W}$ evaluated on the initial loop $(x_l, y_l)$ and expanded to order zero in $\epsilon$. Explicitly, it reads $H = \frac{4\delta}{\tilde{\alpha}}(\delta + \delta\tilde{c}^2 + \tilde{\alpha}^2\tilde{c}^2)$, where the speed of the traveling wave is $1/A(S) = \tilde{c}(S) + \mathcal{O}(\epsilon)$ and its height is $\tilde{\alpha}(S)$. Hence, we conclude that the relations (3) and (4) hold, in the case when $\delta$ is a function of $S$, and that $H$ is constant, to order $\epsilon$. This is a consequence of the fact that the energy is conserved when the cross section varies with the length. The analysis of (3) and (4) reveals two possible behaviors as $\delta \to 0$: either, the loop shrinks size and the velocity is finite or the velocity of the loop becomes unbounded and the height finite. However, since we assume that the initial loop is subsonic ($c_i \ll 1$) and of height larger than the cross section ($\alpha_i > 4$), the only physical solutions have unbounded velocity as the section tapers to zero. Close to $\delta = 0$, an asymptotic solution for the speed and height can be found:

$$c = \frac{\delta^{-1/2}}{2\alpha_i}\left[2 + c_i^2(\alpha_i^2 - 1) + \frac{\delta}{2\alpha_i^2}[2 - c_i^2(\alpha_i^2 - 9)] + O(\delta^2)\right],$$

$$\alpha = \alpha_i(1 + 2c_i^2) + \frac{2\delta}{\alpha_i}(2c_i^2 - 1) + O(\delta^2).$$

A typical plot of $c$ and $\alpha$ as a function of the radius $R$ is shown in Fig. 3. The loop size remains almost constant whereas the velocity blows up. The loop reaches supersonic material velocity for $R(s) = R_0/\alpha_i + O(c_i^2)$.

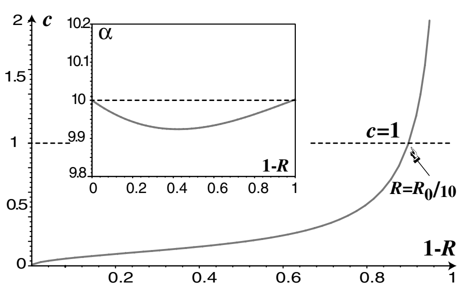


FIG. 3. The loop speed $c$ and height $\alpha$ (see inset) as a function of the radius $R$ $[R(0) = 1,\, c_i = 0.01,\, \alpha_i = 10]$. The loop speed reaches material supersonic velocity when the radius shrinks by a factor of 10.

The previous analysis demonstrates that tapering causes the acceleration of a traveling loop on an infinite filament. The velocity of the loop becomes unbounded and the height finite. When a whip is cracked in the air, the situation is different due to the finiteness of the filament and the conditions at the free end. In particular, since there is no applied tension at the free end, the assumption that the tension is constant throughout the rod is only approximately satisfied for a short period of time when the loop is still close to the handle. The boundary conditions at the whip tip, $s = L$, are easily enforced by requiring that force and curvature vanish identically; that is, $\varphi'(L, t) = F(L, t) = G(L, t) = 0$ for all time $t$. Different boundary conditions at the handle, $s = 0$, can be considered. In order to analyze the role of the free end boundary conditions in the acceleration of the initial impulse, we first use conditions that preserve the initial energy imparted in the whip and assume that once the loop is sent, the handle and the whip inclination remains immobile so as not to produce any additional work: $\varphi(0, t) = x(0, t) = y(0, t) = 0$.

The numerical analysis is performed by modifying a dedicated scheme for the dynamics of planar rods due to Falk and Xu [17] in order to incorporate the effect of varying cross sections and free boundary conditions. The space interval $[0, L]$ is divided into $N$ equally spaced mesh points and time is discretized into steps of duration $\tau$, with $\tau N/L$ set equal to $1/2$, below the limit of the classical Courant-Friedrichs-Lewy condition for stability. The method preserves a discrete version of the energy $H$. At each time step, a semilinear equation for the discretized angle $\varphi$ is solved iteratively and the tension at $s = 0$ is found. This is the force that the hand must apply for the handle to remain motionless. A loop solution $(x_l, y_l)$ is used as initial data with initial velocity and height $(\alpha_i, c_i)$ and initial tension at $s = 0$ given by (4) with $\delta = 1$. A typical evolution of the whip is shown in Fig. 4. As the loop travels to the end of the whip, the tensionless tip is pulled by the loop and rotates around a fixed point. The net effect of this rotary motion is to accelerate the tip of the whip. To identify the different mechanical contributions that cause the loop to accelerate, we introduce the acceleration factor $\gamma = c_{\max}/(2c_i)$, where $c_{\max}$ is the maximal velocity of a

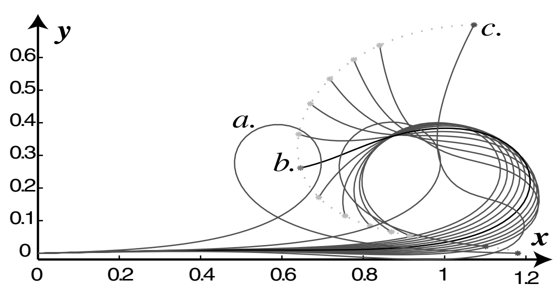


FIG. 4. Numerical solutions of a realistic whip ($L = 2$ m, $c_i = 0.5$, $\alpha_i = L/10$). ($a$) Initial loop; ($b$) whip when the tip reaches the fastest velocity; ($c$) whip at time $t = 0.02$ s.

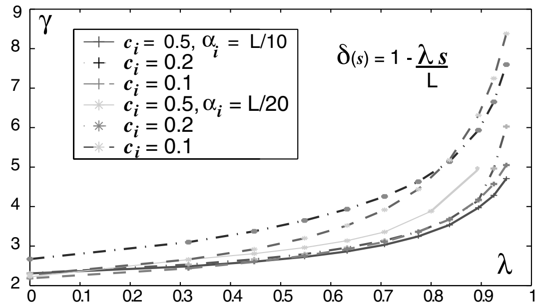


FIG. 5. The acceleration factor $\gamma$ as a function of the tapering.

material point ($\gamma = 1$ for the initial traveling loop). For a nontapered rod, the typical effect of the free end is to accelerate the tip by a factor $\gamma$ between 2 and 3 (see Fig. 5 at $\lambda = 0$). The additional effect of tapering is to enhance $\gamma$ up to 9 when the cross section tapers to $\lambda = 1/20$ of its initial area.

In actual whip cracking, additional energy is provided throughout the entire maneuver by pulling the handle in the opposite direction from the loop. This added tension is the principal cause of loop acceleration in the downward snap and is also used in the forward crack. It corresponds to the choice of boundary conditions $\varphi(0,t) = 0, F(0,t) = \beta, G(0,t) = 0$. To quantify this effect, we measure $\gamma$ for increasing values of $\beta$ and found that $\gamma$ depends on $\beta$ almost linearly. For realistic whips ($L = 2$ m), values of initial velocity ($c_i = 0.1$), tension (7 kN), and tapering ($\lambda = 0.9$) indicate that the tip can reach a velocity 32 times bigger than the initial velocity of the whip, easily reaching supersonic velocities.

Anyone who picks up a whip knows that making it crack requires skill, practice, and dexterity and that mishandling can result in painful experiences. At the physical level, the dangers of mishandling the problem are not as drastic but a simplistic approach based on a single conservation principle fails to provide insight on a beautiful phenomenon. Clearly, the acceleration of an initial impulse on an elastic rod is not easy to achieve and requires the subtle combination of different effects: the formation of an initial traveling loop, the free-end boundary conditions, the tapering, and the added tension.

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